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Relativistic Simultaneity and Causality

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We analyzed two types of relativistic simultaneity associated to an observer: the spacelike simultaneity, given by Landau submanifolds, and the lightlike simultaneity given by past-pointing horismos submanifolds. We study some geometrical conditions to ensure that Landau submanifolds are spacelike and we prove that horismos submanifolds are always lightlike. Finally, we establish some conditions to guarantee the existence of foliations in the space-time whose leaves are these submanifolds of simultaneity generated by an observer. These foliation structure allows us to incorporate the simultaneity submanifolds for studying some dynamical systems, for instance free elementary massless particles.

KEY WORDS: simultaneity; relativity; foliation; causality.

1. INTRODUCTION

It is well known that some problems related with simultaneity have not been solved yet. A number of works treat the local character of relativistic simultaneity accepting that Landau submanifolds (Olivert, 1980) generated by an observer are leaves of a spacelike foliation. However, the fulfillment of this property cannot be ensured on any neighborhood without assuming some additional geometrical conditions. Therefore, when working on a neighborhood where this property does not hold, some difficulties in setting a successful dynamical study arise because each Landau submanifold depends on position p and 4-velocity in p. The study of some of these conditions was the main objective of this study.

In this work, we consider two types of simultaneities: *spacelike simultaneity*, which describes those events that are simultaneous in the local inertial proper system of the observer, and *lightlike simultaneity*, which describes those events which the observer perceives as simultaneous although they are not simultaneous in the their local inertial proper system. The sets of spacelike simultaneous points

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and lightlike simultaneous points determine the Landau submanifold and the past-pointing horismos (Beem and Ehrlich, 1981) submanifolds respectively.

Our next concern is the causality related to these types of simultameity, because we should be able to guarantee, for instance, that Landau submanifolds are spacelike in a given neighborhood. For this, we introduce a new concept, the *tangential causality*, more general than causality and we prove that every Landau submanifold is tangentially spacelike, but it is not necessarily spacelike. On the other hand, we prove that every horismos submanifold is tangentially lightlike and as well as lightlike.

In physics, it is usual to work with synchronizable timelike vector fields. In most cases, the leaves of the orthogonal foliation are considered "simultaneity submanifolds." In this work it is proved that given a C^{∞} future-pointing causal curve, *observer*, and a timelike vector tangent to the observer, there exists, on a certain tubular neighborhood of this observer, a synchronizable timelike vector field containing this vector. Moreover, this vector field is orthogonal to a Landau submanifolds foliation. On the other hand, it is also proved that given an observer there exists, a foliation whose leaves are past-pointing horismos submanifolds of points of the observer (in a tubular neighborhood). The foliations structure allows us to extend simultaneity to neighborhoods of the space-time.

2. PRELIMINARY CONCEPTS

In what follows (M, g) will be a four-dimensional lorentzian space-time manifold.

Definition 1. An open neighborhood $\mathcal{N}_0$ of the origin in T_pM is said to be *normal* if the following conditions hold:

(i) the mapping $\exp_p : \mathcal{N}_0 \to \mathcal{N}_p$ is a diffeomorphism, where $\mathcal{N}_p$ is an open neighborhood of p,

(ii) given $X \in \mathcal{N}_0$ and $t \in [0, 1]$ we have that $tX \in \mathcal{N}_0$.

For a given point $p \in M$, an open neighborhood $\mathcal{N}_p$ of p is a *normal neighborhood* of p if $\mathcal{N}_p = \exp_p \mathcal{N}_0$ where $\mathcal{N}_0$ is a normal neighborhood of the origin in T_pM. Finally, an open set $\mathcal{V} \neq \emptyset$ in M, which is a normal neighborhood of each one of its points, is a *convex normal neighborhood*.

These neighborhoods are useful to obtain a dynamical study of simultaneity. Moreover, the *Whitehead Lemma* asserts that given $p \in M$ and a neighborhood $\mathcal{U}$ of p there exists a simple convex neighborhood $\mathcal{V}$ of p such that $\mathcal{V} \subset \mathcal{U}$ (Sachs and Wu, 1977). Therefore, we can consider these kind of neighborhoods without any loss of generality.

We are going to introduce two static ways to analyze simultaneity: Landau and horismos submanifolds. Given $u \in T_pM$ and the metric tensor field g, we consider the submersion $\Phi : \mathcal{N}_p \to R$ given by $\Phi(q) = g(\exp_p^{-1} q, u)$. The fiber

$$L_{p,u} := \Phi^{-1}(0) \tag{1}$$

is a regular three-dimensional submanifold, called *Landau submanifold*. The next result (Olivert, 1980) guarantees its uniqueness:

Theorem 1. *Given $u \in T_p M$ a future-pointing timelike vector and S_u its orthogonal 3-space, there exists a unique regular three-dimensional submanifold $L_{p,u}$ such that S_u is tangent to $L_{p,u}$ at p and whose points are simultaneous with p in the local inertial proper system of p.*

On the other hand, defining the submersion $\varphi : \mathcal{N}_p - \{p\} \to R$ given by $\varphi(q) = g(\exp_p^{-1} q, \exp_p^{-1} q)$, the fiber

$$E_p := \varphi^{-1}(0) \tag{2}$$

is a regular three-dimensional submanifold, called *horismos submanifold* of p, which has two connected components (Sachs and Wu, 1977). We will call *past-pointing (*respectively *future-pointing) horismos submanifold* of p, E_p^- (resp. E_p^+), to the connected component of (2) in which, for each point $q \in \mathcal{N}_p - \{p\}$, the preimage $\exp_p^{-1} q$ is a past-pointing (respectively future-pointing) lightlike vector.

The points in a Landau submanifold $L_{p,u}$ are simultaneous with p in the local proper system of p (i.e. they are synchronous with p). The points in a past-pointing horismos submanifold E_p^- are observed simultaneously by p, i.e., they belong to light signals which arrive at p simultaneously. On the other hand, the points in a future-pointing horismos submanifold E_p^+ belong to light signals sent from p simultaneously. In general, we will call both, Landau and horismos submanifolds, *simultaneity submanifolds*.

3. TANGENTIAL CAUSALITY AND CAUSALITY

Let p be a point in the space-time M, we introduce the concept of *p-tangential causality*, that it consists, *grosso modo*, in a first-order approximation of the causality on a given neighborhood of p, for two reasons:

(i) Although this concept is more general than causality, it is more operative.
(ii) An observer detects the events of the space-time in its tangent space. So, the p-tangential causality is a kind of "observed causality."

3.1. Tangential Causality

Let V be a four-dimensional vector space regarded as a C^∞ manifold. It can be canonically identified with any of its tangent spaces. For each $v \in V$ there exists a unique isomorphism $\phi_v : T_v V \to V$ such that

$$\bar{\omega}(\phi_v w) = w(\bar{\omega}) \tag{3}$$

for all $w \in T_v V$ and for all $\bar{\omega} \in V^*$.

If (V, g) is a lorentzian vector space, we define a (0, 2)-tensor field $\mathbf{g}$ on TV by

$$\mathbf{g}(w, z) = g(\phi_v w, \phi_v z) \tag{4}$$

where $v \in V$ and $w, z \in T_v V$. Then, $(T_v V, \mathbf{g}\,|_{T_v V})$ is a lorentzian vector space for all $v \in V$ and therefore $(V, \mathbf{g})$ is a lorentzian manifold (Sachs and Wu, 1977).

Applying this result to $T_p M$, for each $v \in T_p M$ we can define a canonical isomorphism ϕ_v of $T_v(T_p M)$ onto $T_p M$. Then $(T_p M, g_p)$ is a lorentzian vector space, where $g_p \equiv g\,|_{T_p M}$. If we define $\mathbf{g}_p$ on $T(T_p M)$ from g_p (according to (4)), then $(T_p M, \mathbf{g}_p)$ is a lorentzian manifold.

Definition 2. Let N be a regular submanifold of M, $p \in N$, $\mathcal{N}_p$ a normal neighborhood of p, and $\mathcal{N}_0 = \exp_p^{-1} \mathcal{N}_p \subset T_p M$. We can consider

$$\exp_p^{-1} N \tag{5}$$

as a regular submanifold in $\mathcal{N}_0$ that we call the p-tangential submanifold of N.

Given $v \in \exp_p^{-1} N \cap \mathcal{N}_0$, we define the p-tangential causality of N at v as the causality of $\exp_p^{-1}$ at v. If this causality is the same at every point of $\exp_p^{-1} N \cap \mathcal{N}_0$ then we define the p-tangential causality of N in $\mathcal{N}_0$ as the causality of (5) at an arbitrary point of $\exp_p^{-1} N \cap \mathcal{N}_0$.

It is easy to prove the next relation

$$T_q N = \exp_{p*v}\left(T_v\left(\exp_p^{-1} N\right)\right), \qquad q = \exp_p v \in \mathcal{N}, \tag{6}$$

and taking $v = 0$, as $\exp_{p*0} \equiv \phi_0$ (Beem and Ehrlich, 1981), we have

$$T_p N = \phi_0\left(T_0\left(\exp_p^{-1} N\right)\right). \tag{7}$$

Then, the causality of N at p coincides with the p-tangential causality of N at the origin. However, given p, q be in $N \cap \mathcal{N}_p$, $v \in \exp_p^{-1} N \cap \mathcal{N}_0$ and $w \in \exp_q^{-1} N \cap \mathcal{N}_0$. The causality of N at p is not necessarily the same as the p-tangential causality of N at $v \neq 0$ or the q-tangential causality of N at w. Nevertheless, if the causality of N at p is nonlightlike then these causalities are locally coincident.

Applying the tangential causality concept to simultaneity submanifolds we obtain the next result:

Proposition 1. *Let $\mathcal{N}_0$ be a normal neighborhood of the origin in T_pM. Then*

(a) *The p-tangential causality of $L_{p,u}$ is spacelike in $\mathcal{N}_0$, for all future-pointing timelike vector $u \in T_pM$.*
(b) *The p-tangential causality of E_p is lightlike in $\mathcal{N}_0 - \{0\}$.*

Proof:

(a) Suppose $v \in \exp_p^{-1} L_{p,u} \cap \mathcal{N}_0$ such that $v \neq 0$, then $g(u, v) = 0$. We define $g_u : \mathcal{N}_0 \to R$ by $g_u(u') = g(u, u')$ for all $u' \in \mathcal{N}_0$. Let $v' \in \mathcal{N}_0$, then v' is in $\exp_p^{-1} L_{p,u}$ if and only if $g_u(v') = 0$. Let $w \in T_v\mathcal{N}_0$, then w is in $T_v(\exp_p^{-1} L_{p,u})$ if and only if $w(g_u) = 0$, i.e., if and only if $\mathbf{g}(\phi_v^{-1}u, w) = 0$ (Sachs and Wu, 1977). Then,
$$T_v\left(\exp_p^{-1} L_{p,u}\right) = \left(\phi_v^{-1}u\right)^{\perp}. \tag{8}$$
But $\phi_v^{-1}u \in T_v\mathcal{N}_0$ is timelike because $\mathbf{g}(\phi_v^{-1}u, \phi_v^{-1}u) = g(u, u) < 0$. Thus (8) is a spacelike subspace and hence $\exp_p^{-1} L_{p,u}$ is a spacelike submanifold in any normal neighborhood $\mathcal{N}_0$ of the origin in T_pM.
(b) An equivalent fact can be found in pages 127 and 128 of Sachs and Wu (1977). $\square$

3.2. Causality of Simultaneity Submanifolds

In a normal neighborhood $\mathcal{N}_p$ of p, the p-tangential causality of a submanifold is not necessarily the same as the causality of this submanifold.

Given $v \in T_pM$ we denote
$$v_q^* := \tau_{pq}v, \quad q \in \mathcal{N}_p, \tag{9}$$
where τ_{pq} is the parallel translation along the unique geodesic segment in $\mathcal{N}_p$ which joins p and q. The vector field v^* on $\mathcal{N}_p$ is said to be *adapted* to the tangent vector v. It is clear that (9) depends differentiably on q and thus it is well defined. Note that the vector field v^* has the same causal character as v because parallel translation keeps causality.

Proposition 2. *Let U be a timelike vector field on an open neighborhood $\mathcal{U}$. U is synchronizable on $\mathcal{U}$ if and only if its orthogonal 3-distribution $(U)^{\perp}$ is a foliation on $\mathcal{U}$. Then, $(U)^{\perp}$ is called* physical spaces 3-distribution *of U and it is denoted by S_U.*

In general, if a submanifold is spacelike or timelike at a point p, there exists a small enough neighborhood of p such that the submanifold is still spacelike or timelike, respectively. Then, a Landau submanifold $L_{p,u}$ is locally spacelike around p, but given any normal neighborhood $\mathcal{N}_p$ of p, the Landau submanifold $L_{p,u}$ is not necessarily spacelike at every point of $\mathcal{N}_p \cap L_{p,u}$. Now we study geometric properties to determine the causality of simultaneity submanifolds:

Proposition 3. *Let $\mathcal{N}_p$ be a normal neighborhood of p.*

(a) *Given future-pointing timelike vector $u \in T_pM$, if u^* (see (9)) is synchronizable in $\mathcal{N}_p$, then $T_qL_{p,u} = (u_q^*)^\perp$ for all $q \in \mathcal{N}_p \cap L_{p,u}$.*
(b) *$T_qE_p = (v_q^*)^\perp$ for all $q \in \mathcal{N}_p \cap E_p$, where $v = \exp_p^{-1} q$.*

Proof:

(a) Let $\mathcal{N}_0 = \exp_p^{-1} \mathcal{N}_p$ be a normal neighborhood of the origin in T_pM and $v = \exp_p^{-1} q$, then $T_v(\exp_p^{-1} L_{p,u}) = (\phi_v^{-1}u)^\perp$ and hence

$$\begin{aligned} T_qL_{p,u} &= \exp_{p*v}\left(T_v\left(\exp_p^{-1} L_{p,u}\right)\right) \\ &= \{\exp_{p*v} w : w \in T_v(T_pM), \times\, \phi_v w \perp u\}. \end{aligned}$$

Let $w \in (\phi_v^{-1}u)^\perp$. Then $g(u^*, (\phi_v w)^*) = 0$ and $g(u^*, v^*) = 0$ because parallel translation keeps orthogonality. Hence $(\phi_v w)^*$ and v^* are in $(u^*)^\perp$. Since u^* is synchronizable in $\mathcal{N}_p$, $(u^*)^\perp$ is a foliation in $\mathcal{N}_p$ and hence $[v^*, (\phi_v w)^*]_q \in (u_q^*)^\perp$. Let us denote $\theta(X)(Y)$ the Lie bracket $[X, Y]$ where X, Y are vector fields. Then $(\theta(v^*)((\phi_v w)^*))_q \in (u_q^*)^\perp$ for all $w \in (\phi_v^{-1}u)^\perp$, i.e., for all $w \in T_v(T_pM)$ such that $(\phi_v w)_q^* \in (u_q^*)^\perp$. Hence, using induction over n

$$(\theta(v^*)^n((\phi_v w)^*))_q \in (u_q^*)^\perp \tag{10}$$

for all $w \in (\phi_v^{-1}u)^\perp$. Since (Helgason, 1962)

$$\exp_{p*v} w = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!}(\theta(v^*)^n((\phi_v w)^*))_q \tag{11}$$

and (10) we obtain $\exp_{p*v} w \in (u_q^*)^\perp$, and since the dimension of $T_qL_{p,u}$ is the same as the dimension of $(u_q^*)^\perp$, we have

$$T_qL_{p,u} = (u_q^*)^\perp.$$

(b) Let $\mathcal{N}_0 = \exp_p^{-1} \mathcal{N}_p$ be a normal neighborhood of the origin in T_pM. Then $T_v(\exp_p^{-1} E_p) = (\phi_v^{-1}v)^\perp$ and hence

$$T_qE_p = \exp_{p*v}\left(T_v\left(\exp_p^{-1} E_p\right)\right) = \{\exp_{p*v} w : w \in T_v(T_pM), \phi_v w \perp v\}.$$

Let $w \in (\phi_v^{-1} v)^\perp$. Then $g(v^*, (\phi_v w)^*) = 0$ and $g(v^*, v^*) = 0$. Hence $(\phi_v w)^*$ and v^* are in $(v^*)^\perp$. We have that (Helgason, 1962) $0 = 2g(\nabla_{(\phi_v w)^*} v^*, v^*)$, and since the torsion vanishes and because $(\nabla_{v^*}(\phi_v w)^*)_q = 0$, we can write

$$\left(\nabla_{(\phi_v w)^*} v^*\right)_q = \left(\nabla_{(\phi_v w)^*} v^*\right)_q - \left(\nabla_{v^*}(\phi_v w)^*\right)_q = -(\theta(v^*)((\phi_v w)^*))_q.$$

Therefore $(\theta(v^*)((\phi_v w)^*))_q \in (v_q^*)^\perp$, for all $w \in (\phi_v^{-1} v)^\perp$, i.e., for all $w \in T_v(T_p M)$ such that $(\phi_v w)_q^* \in (v_q^*)^\perp$. It is easy to show using induction over n that

$$(\theta(v^*)^n((\phi_v w)^*))_q \in (v_q^*)^\perp \tag{12}$$

for all $w \in (\phi_v^{-1} v)^\perp$. Using (11) and (12) we obtain $\exp_{p*v} w \in (v_q^*)^\perp$. Since the dimension of $T_q E_p$ is the same as the dimension of $(v_q^*)^\perp$, we have $T_q E_p = (v_q^*)^\perp$. $\square$

An immediate consequence is the next result:

Corollary 1.

(a) *Given a future-pointing timelike vector $u \in T_p M$, if u^* is synchronizable in $\mathcal{N}_p$, then $L_{p,u}$ is spacelike in $\mathcal{N}_p$.*
(b) *E_p is lightlike in $\mathcal{N}_p - \{p\}$.*

It is important to remark that given a point $p \in M$ and $u \in T_p M$ a future-pointing timelike vector, if the adapted vector field u^* (given by expression (9)) is not synchronizable in a normal neighborhood $\mathcal{N}_p$ of p, then the spacelike causal character of $L_{p,u}$ is not ensured in $\mathcal{N}_p$, but we can always ensure that the p-tangential causality of $L_{p,u}$ is spacelike in $\mathcal{N}_0 = \exp_p^{-1} \mathcal{N}_p$.

There exists a necessary and sufficient condition for Landau submanifolds to be spacelike, but it is less operative than the sufficient condition. Anyway, we are going to enunciate this result without a proof:

Proposition 4. *Let $\mathcal{N}_p$ be a normal neighborhood of p. Given $u \in T_p M$ future-pointing timelike, the following conditions are equivalents:*

(i) *For all $q \in \mathcal{N}_p \cap L_{p,u}$, we have $T_q L_{p,u} = (u_q^*)^\perp$.*
(ii) *There exists a normal neighborhood $\mathcal{N}'_p$ of p such that $\mathcal{N}'_p \cap L_{p,u} = \mathcal{N}_p \cap L_{p,u}$ and u^* is synchronizable in $\mathcal{N}'_p$.*

Given a simultaneity submanifold, we can construct its tangent spaces in any point. Since $u_q^* = \tau_{pq} u$ and parallel translation keeps orthogonality, we have

$(u_q^*)^\perp = \tau_{pq} u^\perp$. So, if u^* is synchronizable in a normal neighborhood $\mathcal{N}_p$ of p and since $u^\perp = T_p L_{p,u}$ we can write (Proposition 3)

$$T_q L_{p,u} = \tau_{pq} T_p L_{p,u}. \tag{13}$$

Analogously, since $v_q^* = \tau_{pq} v$ we have $(v_q^*)^\perp = \tau_{pq} v^\perp$ and hence (Proposition 3)

$$T_q E_p = \tau_{pq} v^\perp. \tag{14}$$

4. SIMULTANEITY FOLIATIONS

On a convex normal neighborhood $\mathcal{V}$ we can define the Landau and horismos submanifolds for each $p \in \mathcal{V}$. In particular, given $\beta : I \to M$ a C^∞ future-pointing causal curve (*observer*), in $\mathcal{V}$ we can define the sets of Landau and horismos submanifolds

$$\{L_{\beta(t)}\}_{t\in I}, \qquad \{E^+_{\beta(t)}\}_{t\in I}, \qquad \{E^-_{\beta(t)}\}_{t\in I},$$

where $L_{\beta(t)}$ denotes $L_{\beta(t),\beta(t)}$ given by (1). Our aim was to study this Landau and horismos submanifolds as leaves of a spacelike and lightlike foliation, respectively.

4.1. Landau Foliations

Theorem 2. *Let $\beta : I \to M$ be an observer and $p = \beta(t_0)$ for any $t_0 \in I$. Then,*

(i) *there exists a convex normal neighborhood $\mathcal{V}$ of p such that $\forall q \in \mathcal{V}$, there exists a unique $t_1 \in I$ such that $\beta(t_1) \in \mathcal{V}$, $q \in L_{\beta(t_1)}$.*

(ii) *there exists a foliation $\mathcal{L}_\beta$ in $\mathcal{V}$ given by*

$$\mathcal{L}_\beta(q) = T_q L_{\beta(t_1)}, \tag{15}$$

where $q \in L_{\beta(t_1)}$, and the leaves of $\mathcal{L}_\beta$ are the Landau submanifolds $L_{\beta(t_1)}$.

The foliation $\mathcal{L}_\beta$ is called Landau foliation *generated by β.*

Proof: There exists an open neighborhood $\mathcal{V}$ of p, that we can consider a convex normal neighborhood, on which the normal exponential map of βI is a diffeomorphism (Sakai, 1996). Therefore, each point in $\mathcal{V}$ is contained precisely in a unique Landau submanifold $L_{\beta(t_1)}$, where $\beta(t_1) \in \beta I \cap \mathcal{V}$. Obviously $L_{\beta(t_1)}$ are leaves of a foliation on $\mathcal{V}$. $\square$

The points of a leaf are simultaneous in the local inertial proper system of the observer. However, given a convex normal neighborhood we can not assure that the Landau submanifolds generated by an observer are leaves of a foliation, because these Landau submanifolds can intersect themselves. In fact, given $\beta : I \to M$ an observer in Minkowski space-time, as the Landau submanifolds

$L_{\beta(t)}$ are hyperplanes orthogonals to the timelike vector $\dot{\beta}(t)$, if β is not a geodesic, the Landau submanifolds generated by β will intersect themselves. Theorem 2 assures us that there exists a small enough neighborhood for Landau submanifolds to be nonintersected between them.

Given the foliation $\mathcal{L}_\beta$ in $\mathcal{V}$, if the adapted vector field (according to (9)) to $\dot{\beta}(t)$ is synchronizable in $\mathcal{V}$ for each $t \in I$ then, Corollary 1 assures that $\mathcal{L}_\beta$ is spacelike in $\mathcal{V}$. Thus, we can build foliations in $\mathcal{V}$ from the orthogonal 3-spaces of the vector fields adapted to $\dot{\beta}(t)$ for each $t \in I$, $(\dot{\beta}(t)^*)^\perp$. Then, the leaf containing $\beta(t)$ is the Landau submanifold $\mathcal{L}_{\beta(t)}$.

Moreover, if $\mathcal{L}_\beta$ is spacelike in $\mathcal{V}$, then it is orthogonal to a timelike 1-foliation. So, we have defined a timelike vector field from β:

Theorem 3. *Let the foliation $\mathcal{L}_\beta$ be in $\mathcal{V}$. If the adapted vector field to $\dot{\beta}(t)$ is synchronizable in $\mathcal{V}$ for each $t \in I$, then $\mathcal{L}_\beta^\perp$ is a synchronizable future-pointing timelike vector field.*

Given the foliation $\mathcal{L}_\beta$ in $\mathcal{V}$, if the vector field adapted to $\dot{\beta}(t)$ is synchronizable in $\mathcal{V}$ for each $t \in I$, by (13) we have that $T_q L_{p,u} = \tau_{pq} T_p L_{p,u}$, where $p = \beta(t_0)$, $u = \dot{\beta}(t_0)$, $t_0 \in I$, and $q \in L_{p,u} \cap \mathcal{V}$. Therefore, because parallel translation keeps orthogonality and $\mathcal{L}_\beta(p)^\perp = u$, we obtain that $\mathcal{L}_\beta(q)^\perp = \tau_{pq} u$.

Moreover, as β is an integral curve of $\mathcal{L}_\beta^\perp$, we can build a different Landau foliation from each integral curve of $\mathcal{L}_\beta^\perp$. These foliations will be the same only in some cases (next result is easy to prove):

Proposition 5. *Let the foliation $\mathcal{L}_\beta$ be in $\mathcal{V}$ and let $\beta' : I' \to M$ be an integral curve of the vector field $\mathcal{L}_\beta^\perp$. If $\mathcal{L}_{\beta'}$ (the Landau foliation generated by β') is well defined and the leaves of $\mathcal{L}_\beta$ (or $\mathcal{L}_{\beta'}$) are totally geodesics, then $\mathcal{L}_{\beta'} = \mathcal{L}_\beta$.*

4.2. Horismos Foliations

Let $\gamma : I \to M$ be a geodesic, $\tilde{\gamma}$ be a positive affine reparametrization of γ (then $\tilde{\gamma}$ is a geodesic). Let $[\gamma]$ denote the corresponding equivalence class of geodesics. Let $\lambda : I \to M$ be a future-pointing lightlike geodesic. Then λ will be called *photon* and the equivalence class $[\lambda]$ will be called *light signal*.

We will study from which points can an observer β receive light signals and to which ones can he send them. The next Proposition can be found in Sachs and Wu (1977):

Proposition 6. *Let $\beta : I \to M$ be a C^∞ future-pointing causal curve and suppose $t_0 \in I$ is given. There exists an open interval $J \subset I$ containing t_0 and a convex normal neighborhood $\mathcal{V}$ of $\beta(t_0)$ such that, for all $x \in \mathcal{V} - \beta J$ there exist*

$t_{-1}, t_1 \in J$ (where $t_{-1} < t_0 < t_1$), a light signal $[\lambda]$ from x to $\beta(t_1)$ and a light signal $[\lambda']$ from $\beta(t_{-1})$ to x. Moreover $t_{-1}, t_1, [\lambda], [\lambda']$ are unique.

Applying Proposition 6 we can construct horismos foliations:

Theorem 4. *Let $\beta : I \to M$ be a C^∞ future-pointing causal curve and $t_0 \in I$. Then,*

(i) *there exists an open interval $J \subset I$ containing t_0 and a convex normal neighborhood $\mathcal{V}^-$ (respectively $\mathcal{V}^+$) of $\beta(t_0)$ such that $\forall q \in \mathcal{V}^- - \beta J$ (respectively $\forall q \in \mathcal{V}^+ - \beta J$) there exists a unique $t_1 \in J$ such that $\beta(t_1) \in \mathcal{V}^-$, $q \in E^-_{\beta(t_1)}$ (respectively $\beta(t_1) \in \mathcal{V}^+$, $q \in E^+_{\beta(t_1)}$).*
(ii) *the mappings*

$$\mathcal{E}^-_\beta : \mathcal{V}^- - \beta J \to \mathcal{P}(\mathcal{V}^- - \beta J) \quad \text{given by} \quad \varepsilon^-_\beta(q) = T_q E^-_{\beta(t_1)},$$

$$\mathcal{E}^+_\beta : \mathcal{V}^+ - \beta J \to \mathcal{P}(\mathcal{V}^+ - \beta J) \quad \text{given by} \quad \varepsilon^+_\beta(q) = T_q E^+_{\beta(t_1)},$$

are foliations. The leaves of $\mathcal{E}^-_\beta$ and $\mathcal{E}^+_\beta$ are the past-pointing and future-pointing horismos submanifolds, respectively

The foliations $\mathcal{E}^-_\beta$ and $\mathcal{E}^+_\beta$ are called respectively past-pointing *and* future-pointing horismos foliation *generated by β.*

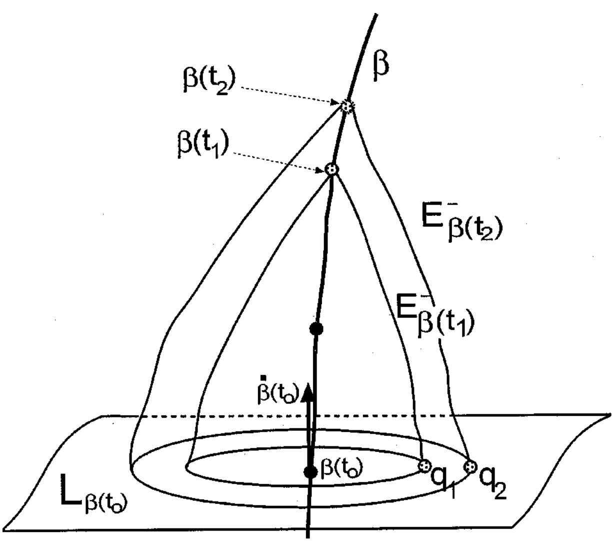

Fig. 1. Scheme of simultaneities on a normal neighborhood.

Given $\mathcal{V}$ a convex normal neighborhood and $\beta : I \to \mathcal{V}$ an observer, we can ensure that the future-pointing (past-pointing) horismos submanifolds generated by β do not intersect themselves.

We can build the neighborhoods $\mathcal{V}^-$ and $\mathcal{V}^+$ of the last Theorem in the following way: We define the open sets

$$\mathcal{V}^- = \left\{p \in \mathcal{V} : p \in E^-_{\beta(t)}, t \in I\right\}, \qquad \mathcal{V}^+ = \left\{p \in \mathcal{V} : p \in E^+_{\beta(t)}, t \in I\right\}.$$

they are convex normal neighborhoods because $\mathcal{V}$ is a convex normal neighborhood.

Hence, there exists a future-pointing horismos foliation generated by β on $\mathcal{V}^+$. Analogously, there exists a past-pointing horismos foliation generated by β on $\mathcal{V}^-$. Moreover, by Corollary 1, this foliations are lightlike foliations. Then, we can define horismos foliations and Landau's foliations on the same convex normal neighborhood (see Fig. 1).

5. CONCLUSION

In this paper we have proved that the Landau submanifolds generated by an observer are not always spacelike leaves of a foliation in a convex normal neighborhood. On the other hand, the horismos submanifolds generated by an observer are always lightlike leaves of a foliation in any convex normal neighborhood. It can be applied to the study of wave fronts of a free massless particle in the following way:

In Symplectic Mechanics, the evolution of a free massless elementary particle can be described by a three-dimensional lightlike foliation Ω in the Minkowski space-time (Souriau, 1970, 1997). This result is generalized (Liern *et al.*, 2000; Liern and Olivert, 1999) to a general space-time M making use of fiber bundles structures that locally preserve the properties of special relativity.

For a spacelike Landau submanifold $L_{p,u}$ and the natural injection $i : L_{p,u} \to M$, the mapping $\mathcal{G} : L_{p,u} \to \mathcal{P}(T(L_{p,u}))$ given by

$$\mathcal{G}(m) = \Omega(m) \cap i_*(T_m L_{p,u}), \quad \forall m \in L_{p,u}, \tag{16}$$

is a two-dimensional foliation (Liern and Olivert, 1995, 1999) whose integral submanifolds can be interpreted as **wave fronts**. Taking into account that we can define a bundle-like (Naveira, 1970) metric on $L_{p,u}$ (Liern and Olivert, 1995), we can prove that the separation between the wave fronts is locally constant (Liern and Olivert, 1995). This result is quite interesting but it is not complete for two reasons:

(i) In these works, authors worked on a unique Landau submanifold,
(ii) and they worked in a neighborhood where the metric was bundle-like.

In this paper we have solved both disadvantages. Firstly, we can interchange Landau submanifolds in a differential way because they form a spacelike foliation when they verify some conditions, as we show in Section 4. Moreover, the second disadvantage is solved by using convex normal neighborhoods in general.

Therefore, given an observer β, we obtain the wave fronts of a free massless elementary particle Ω from the intersection of Ω with the Landau foliation $\mathcal{L}_\beta$ generated by the observer. But, in general, the observed wave fronts are obtained from the intersection of Ω with the past-pointing horismos foliation ε_β^-. This double interpretation of simultaneities makes evident the differences between them.

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